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THE ALBANESE MAPPING FOR A PUNCTUAL HILBERT SCHEME: I. IRREDUCIBILITY OF THE FIBERS

BY

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ABSTRACT. Let f: $X \to A$ be the canonical mapping from an algebraic surface X to its Albanese variety A, $X(n)$ the n-fold symmetric product of X, and H_X^n the punctual Hilbert scheme parameterizing 0-dimensional closed subschemes of length n on X. The latter is a nonsingular and irreducible variety of dimension $2n$, and the "Hilbert-Chow" morphism σ_n: $H_X^n \to X(n)$ is a birational map which desingularizes $X(n)$.

This paper studies the composite morphism

$$\varphi_n: H_X^n \xrightarrow{\sigma_n} X(n) \xrightarrow{f_n} A,$$

where f_n is obtained from f by addition on A. The main result (Part 1 of the paper) is that for $n \gg 0$, all the fibers of φ_n are irreducible and of dimension $2n - q$, where $q = \dim A$. An interesting special case (Part 2 of the paper) arises when $X = A$ is an abelian surface; in this case we show (for example) that the fibers of φ_n are nonsingular, provided n is prime to the characteristic.

Introduction. Let X be an irreducible and nonsingular projective variety of dimension $d > 0$, defined over an algebraically closed ground field. Let f: $X \to A$ be the canonical map from X to its Albanese variety A; over the complex numbers this is the complex torus C^q/L, where L is the lattice generated by the $2q$ periods of a basis of holomorphic 1-forms on X. If $X(n)$ is the n-fold symmetric product of X, whose points represent the 0-cycles on X of degree n, then the map f: $X \to A$ induces by addition on A a map,

$$f_n: X(n) \to A,$$

which is in fact the Albanese mapping of $X(n)$.

In case X is a curve C, the Albanese variety is the Jacobian J of C, and the fibers of f_n: $C(n) \to J$ are projective spaces which represent the complete linear systems of degree n on C (the Abel-Jacobi theorem). The classical Riemann-Roch theorem computes the dimensions of these fibers for all $n > 0$. In particular, Riemann's theorem is the assertion that for $n \gg 0$, all of the fibers of f_n: $C(n) \to J$ have the same dimension $n - g$, where g is the genus of C.

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It is known that the natural generalization of Riemann's theorem to the mapping f_n: $X(n) \to A$ is true where $\dim X = d > 1$. That is, for $n \gg 0$, all of the fibers of f_n are irreducible and of the same dimension $dn - q$, where $q = \dim A$ [10]. In seeking further generalizations (for example, analogues of the full Riemann-Roch theorem) one confronts the fact that $X(n)$ has singularities when $\dim X > 1$, a "defect" not shared by the nonsingular variety $C(n)$. Consequently the fibers of f_n can, and do, acquire singularities which can seriously interfere.

If X is a surface, however, there is available a natural desingularization of $X(n)$. This is the Hilbert scheme H_X^n, an irreducible and nonsingular projective variety of dimension $2n$ whose points represent the closed subschemes of X having Hilbert polynomial n. With each such subscheme Z there is naturally associated a 0-cycle of degree n (a weighted sum of the points in the support of Z); this gives a birational map

$$\sigma_n: H_X^n \to X(n)$$

which is an isomorphism over the smooth locus of $X(n)$, whose points represent the 0-cycles $x_1 + \cdots + x_n$ with n distinct summands. (All these results may be found in [2]. Morphisms analogous to σ_n exist for X of any dimension $d > 0$, but H_X^n is not in general irreducible for $d > 2$ [6]. If $d = 1$, $H_C^n = C(n)$ [7].)

The main purpose of this paper is to prove that when X is a surface and $n \gg 0$, all of the fibers of the composite mapping

$$\varphi_n: H_X^n \xrightarrow{\sigma_n} X(n) \xrightarrow{f_n} A$$

are irreducible and of the same dimension (Corollary 1 of Theorem 3). The proof of this result is the focus of Part 1 of the paper. The necessary preliminaries include a detailed study of the map f_n restricted to the singular locus of $X(n)$, which is carried out for X of any dimension.

Part 2 of the paper treats the special case where X is taken to be an abelian variety A, that is, X equals its own Albanese variety. The principal results of Part 1 (those that follow Theorem 3) are shown to hold for all $n > 0$, rather than just for $n \gg 0$. Moreover, additional information is obtained, notably Corollary 1 of Theorem 6, which states that if A is an abelian surface and n is prime to the characteristic, then all fibers of the mapping φ_n: $H_A^n \to A$ are nonsingular varieties which desingularize the fibers of f_n: $A(n) \to A$. The arguments in Part 2 are based on the group structure of A, and are essentially independent of Part 1.

A forthcoming paper will study the question of smoothness for the mapping φ_n: $H_X^n \to A$, when X is a surface; it can be viewed as a step in the direction of finding a "Roch" theorem to complement the "Riemann" theorem proved here. It should be noted, however, that unlike the case of curves, certain fibers of φ_n can be singular, even in characteristic 0 and for arbitrarily large n; examples will be given.

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Part 1

Remark on terminology. In addition to those established in the introduction, we make the following conventions. A *variety* is a reduced, separated, not necessarily irreducible scheme of finite type over a field k, the latter usually taken to be algebraically closed. By a *point* of V we mean a geometric point, i.e., a map Spec $K \to V$ where K is an algebraically closed extension of the ground field. If V is irreducible, a *generic point* of V is a point localized at the scheme-theoretic generic point of V, and a set $\{v_1, \ldots, v_n\}$ of points of V is a set of *independent generic points* if $(v_1, \ldots, v_n)$ is a generic point of the product V^n.

The fibers of a morphism of varieties are taken to be varieties, *i.e.*, they are always given their reduced structure. The generic fiber refers to the fiber over a generic point as understood above. (In Part 2 we do not need the language of generic points; there, a point is a rational point over the (algebraically closed) ground field.)

Finally, if $v_1 + \cdots + v_n$ is a 0-cycle on V, the associated point of $V(n)$ is written $(v_1 + \cdots + v_n)$.

1. The varieties $X(\pi)$. Until further notice, let X be a projective variety of dimension $d > 0$, irreducible and nonsingular, and let f: $X \to A$ be its Albanese mapping. We assume throughout that dim $A = q > 0$. Of particular interest for us is the nonsmooth locus of $X(n)$ and the behavior of the map f_n: $X(n) \to A$ restricted to the nonsmooth locus. If $(x_1 + \cdots + x_n) \in X(n)$, let $r \leq n$ be the number of distinct x_i. Then, as noted above, $(x_1 + \cdots + x_n)$ is a smooth point of $X(n)$ if and only if $r = n$. We call r the *size* of $(x_1 + \cdots + x_n)$.

Let $\pi = (n_1, \ldots, n_r)$ be a partition of n into r parts, which means each $n_i > 0$ and $\sum_{i=1}^r n_i = n$. We call r the *size* of π and n, following standard usage, the *weight* of π.

To every partition $\pi = (n_1, \ldots, n_r)$ of weight n and size r there corresponds a closed irreducible subvariety $X(\pi)$ of $X(n)$. By definition, $X(\pi)$ is the image of the composite morphism

$$\theta_\pi: X^r \xrightarrow{\Delta n_1 \times \cdots \times \Delta n_r} X^{n_1} \times \cdots \times X^{n_r} = X^n \xrightarrow{\theta_n} X(n), \tag{1}$$

where Δn_i: $X \to X^{n_i}$ is the diagonal map and θ_n is the quotient map (which sends $(x_1, \ldots, x_n)$ to $(x_1 + \cdots + x_n)$). Thus $\theta_\pi(x_1, \ldots, x_r) = (\sum_{i=1}^r n_i x_i)$. Since X^r is complete and irreducible, it follows that its image $X(\pi)$ is a closed irreducible subvariety of $X(n)$.

The following are immediate consequences of the definition:

(2) If $\{x_1, \ldots, x_r\}$ is a set of independent generic points of X, then $(\sum_{i=1}^r n_i x_i)$ is a generic point of $X(\pi)$.

(3) The points of $X(\pi)$ represent the 0-cycles of the form $\sum_{i=1}^r n_i x_i$, where not all the x_i need be distinct.

(4) $\dim X(\pi) = dr$, where $d = \dim X$ and r is the size of π.

We write $f_\pi\colon X(\pi) \to A$ for the restriction to $X(\pi)$ of the morphism $f_n\colon X(n) \to A$. (If, as is standard, we let n represent the partition $(1, \ldots, 1)$ of weight and size n, then $X(n)$, θ_n and f_n are exactly $X(\pi)$, θ_π, and f_π for $\pi = n$, so the notation is consistent.)

The next two sections study the map f_π. In §2 we show that f_π is surjective if the size of π is sufficiently large, and in §3 we consider the fibers of f_π (dimension and irreducibility).

2. Surjectivity of f_π. Let B be an abelian variety and Z, Z' subvarieties of B. We write $Z + Z'$ for the image of the map

$$Z \times Z' \to B \times B \xrightarrow{+} B;$$

the sum $Z + \cdots + Z$ (m summands) is abbreviated mZ. We say Z *generates* B if the inclusion $mZ \hookrightarrow B$ dominates B for $m \gg 0$. As usual we write $m\delta\colon B \to B$ for the isogeny of multiplication by m on B. Then $m\delta(Z)$ is the image of Z under $m\delta$ and should not be confused with mZ.

It is clear that

(5) if Z and Z' are closed (resp. irreducible) subvarieties of B, then so are $Z + Z'$, mZ, and $m\delta(Z)$,

since each of these is the image of a complete (resp. irreducible) variety.

PROPOSITION 1. *Let B be an abelian variety of dimension $p > 0$. Let $Z_1, \ldots Z_s$ be closed irreducible subvarieties of B each of which generates B, with, say,* $\dim Z_1 \geqslant \dim Z_i$ *for* $1 \leqslant i \leqslant s$. *Then for* $s > p - \dim Z_1$, *and all* $n_i > 0$,

$$n_1\delta(Z_1) + \cdots + n_s\delta(Z_s) = B.$$

To prove this, we use two lemmas.

LEMMA 1. *If Z is an irreducible subvariety of B that generates B, then so is $n\delta(Z)$ for any $n > 0$.*

PROOF. We have that $n\delta(Z)$ is irreducible by (5). To see $n\delta(Z)$ generates B consider the commutative diagram (m factors)

$$\begin{array}{ccccc} Z \times \cdots \times Z & \hookrightarrow & B \times \cdots \times B & \xrightarrow{+} & B \\ \downarrow & \longleftarrow n\delta \times \cdots \times n\delta \longrightarrow & \downarrow & & \downarrow n\delta \\ n\delta(Z) \times \cdots \times n\delta(Z) & \hookrightarrow & B \times \cdots \times B & \xrightarrow{+} & B \end{array}$$

where m is chosen so that the top row is dominating. Since the rightmost vertical arrow is also dominating, it follows that the bottom row is dominating, whence the lemma. □

LEMMA 2. *Let W be a proper irreducible subvariety of B, and Z an irreducible subvariety of B that generates B. Then* $\dim W < \dim(Z + W)$.

PROOF. Clearly we may assume Z contains the identity element of B, so $W \subseteq Z + W$ and $\dim(W) \leqslant \dim(Z + W)$. In addition, since $\dim(W) = \dim(\overline{W})$ and (as is easily checked)

$$\dim(Z + W) = \dim(\overline{Z} + \overline{W}),$$

we may assume Z and W are closed. Thus $Z + W$ is closed and irreducible (5), and the equality $\dim(W) = \dim(Z + W)$ forces $W = Z + W$, hence

$$W = Z + W = 2Z + W = \cdots = mZ + W.$$

But $mZ = B$ for $m \gg 0$, since Z generates B, so $W = B + W = B$, a contradiction. □

PROOF OF PROPOSITION 1. Since $\dim Z_i = \dim n_i\delta(Z_i)$ for $1 \leqslant i \leqslant s$, Lemma 1 reduces us to proving $Z_1 + \cdots + Z_s = B$. If $Z_1 + \cdots + Z_s \neq B$, by using (5) and Lemma 2 we see

$$\dim B > \dim(Z_1 + \cdots + Z_s) > \dim(Z_1 + \cdots + Z_{s-1}) > \cdots > \dim Z_1.$$

Since the dimension falls by at least one in each of the s steps to Z_1, it follows that

$$\dim B = p \geqslant \dim Z_1 + s,$$

contradicting the choice of s. □

PROPOSITION 2. *Let $\pi = (n_1, \ldots, n_r)$ be any partition of size $r > q - \dim f(X)$, where $q = \dim A$. Then $f_\pi\colon X(\pi) \to A$ is surjective.*

PROOF. Since X is an irreducible projective variety, $f(X)$ is a closed irreducible subvariety of A that generates A. Keeping in mind (3), it is easy to check that

$$\begin{aligned} f_\pi(X(\pi)) &= n_1\delta(f(X)) + \cdots + n_r\delta(f(X)) \\ &= A, \quad \text{by Proposition 1.} \end{aligned}$$

3. Dimension and irreducibility of the fibers of f_π. The following results are proved in [**10**].

(6) If $n \gg 0$, the generic fiber of $f_n\colon X(n) \to A$ is (nonempty and) irreducible. (It suffices to take $n \geqslant g$, where g is the genus of a generic 1-dimensional linear section on X.)

(7) Let n_0 be the least $n > 0$ for which the conclusion of (6) holds. If $n \geqslant n_0 + q$, then every fiber of f_n is irreducible and of dimension $dn - q$.

Theorems 1 and 2 below extend these results to the mapping $f_\pi\colon X(\pi) \to A$.

Theorem 1 (in case X is a surface) plays an important role in the proof of Theorem 3, our main goal. Theorem 2 has no application in this paper, but is included for completeness.

For the proofs of Theorems 1 and 2 we need

LEMMA 3. *Let Y_1, Y_2 and Z be three irreducible varieties of dimensions d_1, d_2, and s, respectively, and let G_1: $Y_1 \to Z$, G_2: $Y_2 \to Z$ be morphisms, where G_1 is assumed dominating. Form the cartesian diagram*:

$$\begin{array}{ccccc} & & Y_1 \times_Z Y_2 & & \\ & pr_1 \swarrow & & \searrow pr_2 & \\ Y_1 & & & & Y_2 \\ & G_1 \searrow & & \swarrow G_2 & \\ & & Z & & \end{array}$$

(a) *Let V be an irreducible subvariety of $Y_1 \times_Z Y_2$, which dominates Y_1 by the first projection, and satisfies* $\dim V \geqslant d_1 + d_2 - s$. *Then*

(i) $\dim V = d_1 + d_2 - s$,

(ii) *V dominates Y_2 by the second projection, and*

(iii) *$\overline{V}$ (the closure of V) is an irreducible component of $Y_1 \times_Z Y_2$.*

(b) *If G_2 is assumed dominating and G_1 has irreducible generic fiber, then there exists a unique irreducible component W of $Y_1 \times_Z Y_2$ that dominates Y_2 by the second projection, and* $\dim W = d_1 + d_2 - s$.

PROOF. (a) Let $v = (v_1, v_2)$ be a generic point of V, where $v_i \in Y_i$, $i = 1, 2$; by hypothesis v_1 is a generic point of Y_1 and $G_1(v_1) = z = G_2(v_2)$ is a generic point of Z. Note that both residue fields $k(v_1)$ and $k(v_2)$ contain $k(z)$, and that $\mathrm{dtr}_k k(v_1) = d_1$, $\mathrm{dtr}_k k(z) = s$, and $\mathrm{dtr}_k k(v_2) \leqslant d_2$ ($\mathrm{dtr}_K L$ denotes degree of transcendence of L/K). It follows easily that $\mathrm{dtr}_k k(v_1, v_2) \leqslant d_1 + d_2 - s$, and in view of our hypothesis on $\dim V$, (i) holds. In turn this implies $\mathrm{dtr}_k k(v_2) = d_2$, so that v_2 is a generic point of Y_2, and (ii) holds. Finally, let V' be an irreducible component of $Y_1 \times_Z Y_2$ containing V; replacing V by V' above shows that $\dim V' = \dim V = d_1 + d_2 - s$, whence $\overline{V} = V'$, and (iii) holds.

(b) Since G_1 and G_2 are dominating, so are the maps pr_1 and pr_2 and thus at least one irreducible component of $Y_1 \times_Z Y_2$ must dominate (the irreducible) Y_2. Let $W_1, \ldots, W_k$ be the set of all such components. Now the generic (geometric) fiber F of pr_2 is irreducible, it being isomorphic to the corresponding irreducible generic fiber of G_1. Since $F \subseteq \bigcup_{i=1}^{k} W_i$, we have $F \subseteq W_c = W$ for some index c. On the other hand, a generic point of each W_i lies in $F \subseteq W$, hence $W_i \subseteq W$, which proves the uniqueness of W. To see $\dim W = d_1 + d_2 - s$, note that W dominates Y_2 (of dimension d_2) with generic fiber F of dimension $d_1 - s$. $\square$

Theorems 1 and 2 describe the fibers of the mapping $f_\pi\colon X(\pi) \to A$, given hypotheses on the partition π. A key idea in the proof of both theorems is to interpret the fiber $f_\pi^{-1}(a)$ $(a \in A)$ as a fiber product of two smaller varieties (and then to apply Lemma 3) as follows: Break up $\pi = (n_1, \ldots, n_r)$ into subpartitions $\pi_1 = (n_1, \ldots, n_s)$, $\pi_2 = (n_{s+1}, \ldots, n_r)$, and form the diagram with cartesian square:

$$\begin{array}{ccccc} & X(\pi_1) \underset{A}{\times} X(\pi_2) & \xrightarrow{\ \alpha\ } & X(\pi) \\ {}^{pr_1}\swarrow & & \searrow^{pr_2} & \\ X(\pi_1) & & X(\pi_2) & \\ {}_{f_{\pi_1}}\searrow & & \swarrow_{a - f_{\pi_2}} & \\ & A & & \end{array} \tag{8}$$

where $a - f_{\pi_2}$ is the map given by composing f_{π_2} with inversion and then translation by a, and α is the map that "adds 0-cycles." One checks easily that the image of α in $X(\pi)$ is the fiber $f_\pi^{-1}(a)$.

THEOREM 1. *Let m_0 be the least $m > 0$ such that $f_\pi\colon X(\pi) \to A$ is surjective whenever size $(\pi) \geqslant m$. Then for any partition π of size $r \geqslant m_0 + q$, every fiber of f_π is equidimensional of dimension $dr - q$, where $d = \dim X$, $q = \dim A$.*

REMARK. According to Proposition 2, $m_0 \leqslant q - \dim f(X) + 1$.

PROOF. Let a be a point of A and let U be an irreducible component of the fiber $f_\pi^{-1}(a)$. By extending the ground field, we may assume that X, A and U are all defined over the algebraically closed ground field K, and that a is K-rational. We have that $\dim U \geqslant dr - q$ by the theorem on the dimension of fibers of a morphism [13, p. 60]. We must prove this inequality is an equality.

Let $u = (\sum_{i=1}^r n_i x_i)$ be a generic point of U, where $\pi = (n_1, \ldots, n_r)$, as in (3). We first claim that

(9) after renumbering the x_i (and n_i accordingly), $\{x_1, \ldots, x_{r-q}\}$ is a set of independent generic points of X over K.

To prove this, note that the map $\theta_\pi\colon X^r \to X(\pi)$ which defined $X(\pi)$ above (1) induces an inclusion of residue fields

$$K(u) \to K(x_1, \ldots, x_r),$$

since $\theta_\pi((x_1, \ldots, x_r)) = (\sum_{i=1}^r n_i x_i) = u$. Since $K(u)$ is the function field of U, we see that

$$\operatorname{dtr}_K K(x_1, \ldots, x_r) \geqslant \operatorname{dtr}_K K(u) = \dim U \geqslant dr - q.$$

But $K(x_1, \ldots, x_r)$ is the top of the tower

$$K \subseteq K(x_1) \subseteq K(x_1, x_2) \subseteq \cdots \subseteq K(x_1, \ldots, x_r),$$

and since $\operatorname{dtr}_K K(x_i) \leqslant d$ for all i, at least $r - q$ of the steps in the tower must raise the dtr by the amount d, otherwise $\text{dtr}_K K(x_1, \ldots, x_r) < dr - q$. Renumbering the x_i, we may suppose these to be the first $r - q$ steps, and we obtain $\text{dtr}_K K(x_1, \ldots, x_{r-q}) = d(r - q)$, which is equivalent to (9).

Assuming this renumbering accomplished, break up π as follows: $\pi = (\pi_1, \pi_2)$, with $\pi_1 = (n_1, \ldots, n_{r-q})$ and $\pi_2 = (n_{r-q+1}, \ldots, n_r)$, and form the diagram (8). Let $v_1 = (\Sigma_{i=1}^{r-q} n_i x_i)$ and $v_2 = (\Sigma_{i=r-q+1}^{r} n_i x_i)$, and let V be the closed irreducible subvariety of $X(\pi_1) \times_A X(\pi_2)$ having generic point (v_1, v_2). Clearly V dominates U under the mapping α (since $\alpha(v_1, v_2) = u$), so $\dim V \geqslant \dim U \geqslant dr - q$. We now show, using (a) of Lemma 3, that $\dim V = dr - q$, which completes the proof. The last inequality already checks the hypothesis required on $\dim V$; it remains to check that V dominates $X(\pi_1)$ and that $X(\pi_1)$ dominates A. But by construction $v_1 = (\Sigma_{i=1}^{r-q} n_i x_i)$ is a generic point (2) of $X(\pi_1)$, so V dominates $X(\pi_1)$, and $X(\pi_1)$ dominates A since π_1 has size $r - q \geqslant m_0$. Now Lemma 3, part (a), (i) shows that $\dim V = dr - q$, and Theorem 1 is proved. □

Let r_0 be the least $r > 0$ such that for any partition π of size $\geqslant r$, the map f_π is surjective and has all fibers equidimensional of dimension $dr - q$. Then by Theorem 1,

(10) r_0 exists and satisfies $r_0 \leqslant 2q - \dim f(X) + 1$.

Proposition 3. *If $n > r_0$, then every irreducible component of every fiber of f_n: $X(n) \to A$ meets the smooth locus of $X(n)$.*

Proof. Let U be a component of $f_n^{-1}(a)$. If U contains no smooth points of $X(n)$ (points of size n), then clearly $U \subseteq X(\pi) \cap f_n^{-1}(a) = f_\pi^{-1}(a)$, where $\pi = (1, 1, \ldots, 1, 2)$ is the partition of weight n and size $n - 1$. Since $n > r_0$, $\dim U = dn - q$. On the other hand, since $n - 1 \geqslant r_0$, $f_\pi^{-1}(a)$ has dimension $d(n - 1) - q$, contradicting $U \subseteq f_\pi^{-1}(a)$. □

Theorem 2. *Suppose that f_m: $X(m) \to A$ is surjective and has irreducible generic fiber. Let π be a partition of size r for which at least $m + q$ of the parts equal* 1. *Then every fiber of f_π: $X(\pi) \to A$ is irreducible and of dimension $dr - q$.*

Remarks. For the partition $(1, \ldots, 1)$ of weight and size $n \geqslant m + q$, Theorem 2 is just (7). According to (6), the hypothesis on f_m is verified for $m \geqslant g$, where g is the genus of a generic curve of X.

Proof. Let a be a point of A and let U be an irreducible component of the fiber $f_\pi^{-1}(a)$. We may assume that X, A, and U are all defined over the same algebraically closed ground field K and that a is K-rational. We have that $\dim U \geqslant dr - q$ by the theorem on the dimension of fibers.

Write $\pi = (n_1, \ldots, n_r)$ and let $u = (\Sigma_{i=1}^{r} n_i x_i)$ be a generic point of U. Call n_i the associated part of x_i. We claim that we can choose a subset of m of the

x_i, all having associated part 1, to form a set of independent generic points of X over K.

First note that, according to (9), we can choose from among the x_i a set $S = \{x_{i_1}, \ldots, x_{i_{r-q}}\}$ of independent generic points of X over K. Now at least m of the $x_i \in S$ must have associated part 1, since by hypothesis at least $m + q$ of the x_i have associated part 1, and only q of the x_i are not in S. By renumbering the x_i and n_i, therefore, we may assume that $(x_1, \ldots, x_m)$ is a set of independent generic points with associated parts $n_1 = n_2 = \cdots = n_m = 1$.

Now break up $\pi = (\pi_1, \pi_2)$ with $\pi_1 = (1, 1, \ldots 1)$ (m parts), $\pi_2 = (n_{m+1}, \ldots n_r)$, and form the diagram (8). Let $v_1 = (\Sigma_{i=1}^{m} n_i x_i)$ and $v_2 = (\Sigma_{i=m+1}^{r} n_i x_i)$, and let V be the closed irreducible subvariety of $X(\pi_1) \times_A X(\pi_2)$ having generic point (v_1, v_2). Arguing as in the proof of Theorem 1, we see that V covers U under α and that the hypotheses of Lemma 3, part (a), are satisfied; consequently:

(i) $\dim V = dr - q$ ($\Rightarrow \dim U = dr - q$),

(ii) V dominates $X(\pi_2)$, and

(iii) V is an irreducible component of $X(\pi_1) \times_A X(\pi_2)$.

On the other hand, the hypotheses of Lemma 3, part (b) are also satisfied for (8), namely: $f_{\pi_1} = f_m$ is surjective and has irreducible generic fiber by hypothesis, and since f_{π_2} is surjective (size $(\pi_2) = r - m \geqslant q$), so also is $a - f_{\pi_2}$. Thus there is a unique irreducible component W of $X(\pi_1) \times_A X(\pi_2)$ dominating $X(\pi_2)$. By (ii) and (iii) above, $V = W$; thus W covers U under α. If U' were another component of $f_\pi^{-1}(a)$, we could repeat the above argument for U' and thereby conclude that W covers U', whence $U = U'$. Thus $f_\pi^{-1}(a)$ is irreducible (and of dimension $dr - q$), and Theorem 2 is proved. □

EXAMPLE. Let $X = C$, a curve of genus $g \geqslant 1$, and let J be its Jacobian. Let $n = cr$ with $r \geqslant g$, $c > 1$, and let $\pi = (c, \ldots, c)$ (r parts), a partition of weight n and size r. (Thus π does not satisfy the hypothesis of Theorem 2.) Let i: $C(r) \to C(\pi)$ be the map sending $(x_1 + \cdots + x_r)$ to $(cx_1 + \cdots + cx_r)$, evidently a bijective morphism. Consider the commutative diagram:

$$\begin{array}{ccc} C(r) & \xrightarrow{i} & C(\pi) \\ f_r \downarrow & & \downarrow f_\pi \\ J & \xrightarrow[c\delta]{} & J \end{array} .$$

Now the fibers of f_r are projective spaces, and the map $c\delta$ is finite to one. Thus the fibers of f_π are (set theoretically) finite disjoint unions of projective spaces, and in particular are not irreducible.

The end of [10] consists of two examples relevant to statements (6) and (7).

4. Irreducibility of the fibers of φ_n. We now restrict X to be a (projective, irreducible, and nonsingular) surface, and let H_X^n be the Hilbert scheme representing 0-dimensional closed subschemes of length n on X. As discussed in the introduction, there is a birational map $\sigma_n \colon H_X^n \to X(n)$, which is an isomorphism over the smooth locus of $X(n)$, and which desingularizes $X(n)$. We are out to show that for $n \gg 0$, all fibers of the composite mapping

$$\varphi_n \colon H_X^n \xrightarrow{\sigma_n} X(n) \xrightarrow{f_n} A$$

(which is the Albanese mapping of H_X^n) are irreducible and of dimension $2n - q$; this is a corollary of Theorem 3. For the proof of Theorem 3 we need, in addition to the work above on the morphisms $f_\pi \colon X(\pi) \to A$, the following fact concerning the behavior of σ_n over the singular locus:

(11) If $(x_1 + \cdots + x_n) \in X(n)$ has size r, then $\dim \sigma_n^{-1}(x_1 + \cdots + x_n) = n - r$ [5, p. 820].

THEOREM 3. *Let X be an irreducible, nonsingular, and projective algebraic surface, defined over an algebraically closed field of any characteristic. If $n > r_0 + q$ (where r_0 is as in* (10)), *then*

(a) *every fiber of the Albanese mapping*

$$\varphi_n \colon H_X^n \xrightarrow{\sigma_n} X(n) \xrightarrow{f_n} A$$

is equidimensional of dimension $2n - q$, and

(b) *for all $a \in A$, the mapping $V \mapsto \sigma_n(V)$ is a bijection from the set of irreducible components of $\varphi_n^{-1}(a)$ to the set of irreducible components of $f_n^{-1}(a)$. Moreover, V and $\sigma_n(V)$ are birationally equivalent under σ_n.*

PROOF. We remark that the hypothesis $n > r_0 + q$ implies, in particular, that f_n and (therefore) φ_n are surjective.

Let a be any point of A, and V an irreducible component of $\varphi_n^{-1}(a)$. By the theorem on the dimension of fibers, $\dim V > 2n - q$.

Since V is irreducible, its image $\sigma_n(V)$ is an irreducible subvariety of some component U of $f_n^{-1}(a)$. We shall prove that $\sigma_n(V) = U$.

Let v be a generic point of V, and suppose that $\sigma_n(v) = (\sum_{i=1}^r n_i x_i)$ is a point of size r, where $r < n$. Then clearly

$$\sigma_n(V) \subseteq X(\pi) \cap f_n^{-1}(a) = f_\pi^{-1}(a),$$

where π is the partition $(n_1, \ldots, n_r)$. We claim that $r > n - q$.

To prove the claim, first note that the theorem on the dimension of fibers applied to $\sigma_n | V$ shows

$$\begin{aligned} 2n - q < \dim V &= \dim \sigma_n(V) + \dim(\sigma_n | V)^{-1}(\sigma_n(v)) \\ &< \dim \sigma_n(V) + n - r, \end{aligned} \tag{12}$$

where the second line follows from (11). Now using in (12) the crude estimate

$$\dim \sigma_n(V) \leqslant \dim f_\pi^{-1}(a) \leqslant \dim X(\pi) = 2r,$$

we see that

$$2n - q \leqslant \dim V \leqslant 2r + n - r,$$

thus $n - q \leqslant r$, as claimed.

Since π has size $r \geqslant n - q \geqslant r_0$, we have that $f_\pi^{-1}(a)$ is equidimensional of dimension $2r - q$. This yields the refined estimate

$$\dim \sigma_n(V) \leqslant \dim f_\pi^{-1}(a) = 2r - q;$$

using this in (12) instead of $2r$ gives

$$2n - q \leqslant n + r - q,$$

which shows that $n \leqslant r$, therefore $n = r$, and $\dim \sigma_n(v) \leqslant 2n - q$. Consequently (again using (12)),

$$\dim V = \dim \sigma_n(V) = 2n - q,$$

which proves (a).

Moreover, every fiber of f_n: $X(n) \to A$ is equidimensional of dimension $2n - q$ since $n > r_0$. Thus $\dim U = 2n - q$, and since $\sigma_n(V)$ is an irreducible subvariety of U of dimension $2n - q$, we conclude that $\sigma_n(V) = U$, *i.e.*, $\sigma_n(V)$ is an irreducible component of the fiber $f_n^{-1}(a)$.

It remains to prove (b), which asserts the bijectivity of the map $V \mapsto \sigma_n(V) = U$, and the birationality of V and U under σ_n. But by Proposition 3 (which applies since $n > r_0$) every component of $f_n^{-1}(a)$ meets the open set where σ_n is an isomorphism; from this (b) follows immediately and Theorem 3 is proved. □

Corollary 1. *If $n \gg 0$ (more precisely: if n is such that the conclusions of Theorem* 3 *and* (7) *hold simultaneously), then every fiber of the Albanese mapping*

$$\varphi_n: H_X^n \xrightarrow{\sigma_n} X(n) \xrightarrow{f_n} A$$

is irreducible of dimension $2n - q$, and birationally equivalent to the corresponding fiber of f_n.

Proof. An immediate deduction from (7) and Theorem 3. □

We recall that a variety is called regular if it has trivial Albanese variety. In [8] it is proved that if V is a projective irreducible variety of any dimension, then for $n \gg 0$ every fiber of the map

$$f_n: V(n) \to \mathrm{Alb}(V)$$

is a regular variety. Applying this when $V = X$, we obtain

Corollary 2. *If $n \gg 0$, every fiber of φ_n: $H_X^n \to A$ is a regular variety.*

PROOF. Since being a regular variety is a birational invariant, the corollary follows from Corollary 1. □

Let $g\colon V \to W$ be a dominating morphism of irreducible varieties. We say that a property P holds for the *general fiber* of g if there is a nonempty open subset of W over which all the fibers have the property P. We recall that

(13) the general fiber of g is irreducible if and only if the field extension $k(V)/k(W)$ is primary (*i.e.*, the algebraic closure of $k(W)$ in $k(V)$ is purely inseparable over $k(W)$) [9, p. 90].

Since H_X^n and $X(n)$ are birationally equivalent, the field extensions $k(H_X^n)/k(A)$ and $k(X(n))/k(A)$ are the same. Thus the general fiber of φ_n is irreducible (and birationally equivalent to the corresponding general fiber of f_n) as soon as the general fiber of f_n is irreducible. This is possibly useful in seeking to sharpen the lower bounds on n in the above results, at least in certain cases (*e.g.* Theorem 4(b) of Part 2). The following example shows what can go wrong at the special fiber.

EXAMPLE. Let A be an abelian surface and let X be the surface obtained by blowing up A at its identity e; the general fiber of the Albanese mapping $f\colon X \to A$ is thus a point, and $f_n^{-1}(e) = D$, a projective line. Since the general fiber of f_n is irreducible when $n = 1$, all of the fibers of f_n are irreducible when $n = 1 + 2 = 3$, by (7). However, when $n = 2$, $f_2^{-1}(e)$ consists of two irreducible components of dimension 2: one component U_1 is essentially the fiber of the addition map $A(2) \to A$ over e, and the other component U_2 is $D(2)$. (The general fiber of $f_2\colon X(2) \to A$ is irreducible.)

There are irreducible components V_1, V_2 of $\varphi_2^{-1}(e)$ dominating U_1, U_2 respectively. Since the generic points of U_i have size $n = 2$, it follows that $\sigma_2\colon V_i \to U_i$ is generically 1-1, hence $\dim V_i = 2$ for $i = 1, 2$. But let L be the locus in $U_2 = D(2)$ of points of size 1; clearly $\dim L = 1$ and, by (11), $\dim \sigma_2^{-1}(x) = 1$ for all $x \in L$. Thus there is a two-dimensional locus V_3 in $\varphi_2^{-1}(e)$ dominating L, therefore $\varphi_2^{-1}(e)$ has at least three components. This shows that (b) of Theorem 3 need not be true for low values of n.

PART 2

In Part 2 we study the special case of the Albanese mapping where X is taken to be an abelian variety A of dimension $q > 0$. Then the map $f_n\colon A(n) \to A$ is given by addition (in A) of the points of the 0-cycles, *i.e.*, $f_n((a_1 + \cdots + a_n)) = \sum_{i=1}^n a_i$. We study the fibers of f_n and of the composite mapping

$$\varphi_n\colon H_A^n \xrightarrow{\sigma_n} A(n) \xrightarrow{f_n} A,$$

the latter especially (but not exclusively) in case A is an abelian surface. We obtain strengthened versions of some of the results of Part 1 (with less effort), and new information by exploiting the group structure of A.

Henceforth, if we say v is a point of a variety V, or write $v \in V$, we mean v is a rational point of V (over the algebraically closed ground field), unless otherwise stated. We write μ: $A \times A \to A$ for the group law on A, and μ_a: $A \to A$ for translation by $a \in A$ (i.e., $\mu_a(b) = a + b$). The Hilbert scheme H_A^n is abbreviated to H^n; if Z is a 0-dimensional closed subscheme of length n on A, the associated point of H^n is written $[Z]$.

5. Dimension, irreducibility and regularity of the fibers. We begin by describing natural actions of A on H^n and on $A(n)$ for which the morphism σ_n: $H^n \to A(n)$ is an A-morphism. These actions are key tools for all that follows.

If Z is a 0-dimensional closed subscheme of length n on A, we define its translate Z_a by $a \in A$ to be the pullback

$$\begin{array}{ccc} Z_a & \overset{\mu^*_{-a}}{\to} & Z \\ \downarrow & & \downarrow \\ A & \underset{\mu_{-a}}{\to} & A \end{array}$$

Similarly, if $(a_1 + \cdots + a_n) \in A(n)$, we define its translate by a to be the point $(\mu_a(a_1) + \cdots + \mu_a(a_n))$. The actions we seek are given by the morphisms:

$$\begin{gathered} T_n': H^n \times A \to H^n, \\ ([Z], a) \mapsto [Z_a], \\ T_n: A(n) \times A \to A(n), \\ ((a_1 + \cdots + a_n), a) \mapsto (\mu_a(a_1) + \cdots + \mu_a(a_n)); \end{gathered} \tag{14}$$

which are obtained as follows.

T_n is the morphism induced (by the universal property of a quotient variety [12, pp. 57–60]) by the morphism

$$\begin{gathered} A^n \times A \to A(n), \\ ((a_1, \ldots, a_n), a) \mapsto (\mu_a(a_1) + \cdots + \mu_a(a_n)), \end{gathered}$$

which is invariant under the usual action of the symmetric group Σ_n on the first factor.

To obtain T_n' we must (by the universal property of the Hilbert Scheme [3]) give a closed subscheme W_n' of $A \times H^n \times A = A \times V$, which is finite and flat over V, and has as fiber over $([Z], a) \in V$ the subscheme Z_a. If $W_n \subseteq A \times H^n$ is the universal closed subscheme, one checks that the pullback

$$\begin{array}{ccc} W'_n & \longrightarrow & W \\ \downarrow & \square & \downarrow \\ A \times V & \longrightarrow & A \times H^n \\ (b, [Z], a) & \longmapsto & (b - a, [Z]) \end{array}$$

does the job.

REMARKS–NOTATION. Let H_a^n (resp. F_a^n) denote the fibers over $a \in A$ of the mappings φ_n (resp. f_n), and for all $c \in A$ let $T'_{n,c}$: $H^n \to H^n$ (resp. $T_{n,c}$: $A(n) \to A(n)$) denote the morphisms of translation by c induced by the actions (14). We remark that translation takes fibers to fibers; more precisely, if $a, b \in A$ and c satisfies $nc = b - a$, then $T'_{n,c}|H_a^n$: $H_a^n \to H_b^n$ and $T_{n,c}|F_a^n$: $F_a^n \to F_b^n$ are isomorphisms. In this way A acts transitively on the set of fibers of φ_n (resp. f_n) and the stabilizer of any fiber is the n-torsion group of A. In particular, all the fibers of φ_n (resp. f_n) are isomorphic to one another, so anything true for the general fiber or any special fiber is true for all fibers "by translation."

Theorems 4 and 5 below show (among other things) that the conclusions of the corollaries of Theorem 3 hold for all $n > 0$, not just for $n \gg 0$, when $X = A$ is an abelian surface.

THEOREM 4. (a) *For all $n > 0$, each fiber F_a^n of the mapping f_n: $A(n) \to A$ is irreducible and of dimension $nq - q$, where $q = \dim A$.*

(b) *If A is an abelian surface, then for all $n > 0$, each fiber H_a^n of the mapping*

$$\varphi_n\colon H^n \xrightarrow{\sigma_n} A(n) \xrightarrow{f_n} A$$

is irreducible and of dimension $2n - 2$. Furthermore, H_a^n is birationally equivalent (by restriction of σ_n) to the corresponding fiber F_a^n of f_n.

PROOF. (a) It is enough to prove that the fiber F_e^n over the identity $e \in A$ is irreducible; the rest follows by translation. But F_e^n is the image of the composite map

$$j\colon A^{n-1} \xrightarrow{i} A^n \xrightarrow{\theta_n} A(n), \tag{15}$$

where i is the imbedding sending $(a_1, \ldots, a_{n-1})$ to $(a_1, \ldots, a_{n-1}, -\Sigma_{i=1}^{n-1} a_i)$, and θ_n is the quotient map. Since A^{n-1} is irreducible, so is its image.

(b) Since A is a surface, H^n is irreducible (and nonsingular), and σ_n is a birational map. The argument sketched following (13) shows that (b) holds for the general fiber of φ_n, therefore (b) holds for all fibers by translation. $\square$

THEOREM 5. *Let A be an abelian variety of any dimension. Then for all $n > 0$, every fiber of the mapping f_n: $A(n) \to A$ is a regular variety (i.e. has trivial Albanese variety).*

PROOF. By translation, it is enough to prove this for F_e^n, the fiber over the identity. Let g: $F_e^n \to B$ be a rational mapping to an abelian variety B; we must prove that g is constant.

The surjective morphism j: $A^{n-1} \to F_e^n$ given by (15) yields a diagram

$$\begin{array}{ccc} A^{n-1} & \xrightarrow{j} & F_e^n \\ & H \searrow \quad \swarrow g & \\ & B & \end{array}$$

where H is defined by composition, and it is enough to prove that H is constant.

By the general theory of rational maps of abelian varieties (see, e.g. [1]), after a translation of B, H will be a homomorphism of abelian varieties. Since $H((a_1, \dots, a_{n-1}))$ does not depend on the order of the arguments, all of the homomorphisms

$$h_\nu: A \xrightarrow{i_\nu} A^{n-1} \xrightarrow{H} B,$$

where i_ν $(1 < \nu < n - 1)$ is the standard injection to the νth factor, are equal. Writing h for any of the h_ν, we have

$$H((a_1, \dots, a_{n-1})) = \sum_{i=1}^{n-1} h(a_i)$$

for all $(a_1, \dots, a_{n-1}) \in A^{n-1}$.

But from (15) it is clear that $(a_1, \dots, a_{n-1})$ and $(a_1, \dots, a_{n-2}, -\sum_{i=1}^{n-1} a_i)$ have the same image in F_a^n under j, thus

$$H((a_1, \dots, a_{n-1})) = H\left(\left(a_1, \dots, a_{n-2}, -\sum_{i=1}^{n-1} a_i\right)\right), \quad \text{i.e.,}$$

$$\sum_{i=1}^{n-1} h(a_i) = \sum_{i=1}^{n-2} h(a_i) - \sum_{i=1}^{n-1} h(a_i)$$

which becomes

$$\sum_{i=1}^{n-2} h(a_i) + 2h(a_{n-1}) = e.$$

Fixing $a_2, \dots, a_{n-1}$ and letting a_1 vary, we conclude that h and (thus) H are constant, which proves the theorem. □

COROLLARY. *Let A be an abelian surface. Then for any $n > 0$, every fiber of the map φ_n: $H^n \to A$ is a regular variety.*

PROOF. Being a regular variety is a birational invariant, so the corollary follows from Theorems 4 and 5. □

6. Normality and nonsingularity of the fibers. As remarked earlier, A acts by translation on the fibers of φ_n and f_n transitively but not simply; the stabilizer of any fiber is the n-torsion group of A. Apparently the fibrations $\varphi_n\colon H^n \to A$ and $f_n\colon A(n) \to A$ are "twisted products" of the base and a fiber, which one wants to "untwist" into simple products. To do this, form the cartesian diagrams

$$\begin{array}{ccc} A(n) & \xleftarrow{(n\delta)^*} & A(n)_1 \\ {\scriptstyle f_n}\downarrow & \square & \downarrow \\ A & \xleftarrow[n\delta]{} & A_1 \end{array} \qquad \begin{array}{ccc} H^n & \xleftarrow{(n\delta)^*} & H_1^n \\ {\scriptstyle \varphi_n}\downarrow & \square & \downarrow \\ A & \xleftarrow[n\delta]{} & A_1 \end{array} \tag{16}$$

where $A_1 = A$ and $n\delta$ is the isogeny given by multiplication by n. We now show that the fibrations of $A(n)_1$ and H_1^n over A_1 are actually products.

THEOREM 6. *There exist A_1-isomorphisms*

$$\lambda'_n\colon H_1^n \to H_e^n \times A_1, \qquad \lambda_n\colon A(n)_1 \to F_e^n \times A_1,$$

where e is the identity of $A = A_1$.

PROOF. We write down λ'_n and its inverse using the action T'_n (14); one obtains λ_n and its inverse analogously. We note that as point sets,

$$H_1^n = \{([Z], a) | \varphi_n([Z]) = na\},$$

and

$$H_e^n \times A_1 = \{([Z], a) | \varphi_n([Z]) = e\}.$$

Let λ'_n be the composite morphism

$$\lambda'_n\colon H_1^n \hookrightarrow H^n \times A_1 \xrightarrow{1\times -\delta} H^n \times A_1 \xrightarrow{(T'_n, pr_2)} H^n \times A_1 \xrightarrow{1\times -\delta} H^n \times A_1$$

which on points satisfies

$$\lambda'_n([Z], a) = ([Z_{-a}], a) \in H_e^n \times A_1.$$

The natural candidate for the inverse is the composite morphism

$$H_e^n \times A_1 \hookrightarrow H^n \times A_1 \xrightarrow{(T'_n, pr_2)} H^n \times A_1,$$

which on points satisfies $([Z], a) \mapsto ([Z_a], a) \in H_1^n$.

One checks easily that the composition of these morphisms in either order is the identity, so the theorem is proved. □

In order to exploit Theorem 6, we need the following two facts. Let V and W be irreducible varieties defined over an algebraically closed field.

(17) The product $V \times W$ is normal (resp. nonsingular) if and only if V and W are each normal (resp. nonsingular).

(18) If $g\colon V \to W$ is an etale map, then W normal (resp. nonsingular) implies V normal (resp. nonsingular). (The converse holds if g is surjective.)

Proof. (17) If V and W are normal, then so is $V \times W$, by [**9**, p. 148]. Conversely, let V' and W' be the normalizations of V and W, and consider the finite and birational map $V' \times W' \to V \times W$ defined in the obvious way. Since $V \times W$ is normal, Zariski's Main Theorem implies this map is bijective, and hence (ZMT again) an isomorphism, whence $V = V'$, $W = W'$. Statement (17) for "nonsingular" is proved in [**9**, p. 199] (and is not difficult).

(18) These assertions are proved in [**4**, Exp. 1, 9.2, 9.10].

Let p be the characteristic of the ground field. It is well known that the isogeny $n\delta$ is etale if and only if $p \nmid n$ [**11**, pp. 64, 74]. Since the pullback of an etale morphism is etale, we have that the morphisms $(n\delta)^*$ of (16) are etale if $p \nmid n$. This leads to Corollaries 1 and 2.

Corollary 1. *Let A be an abelian surface, and let $p \nmid n > 0$. Then every fiber H^n_a of the map $\varphi_n\colon H^n \to A$ is a nonsingular variety. Moreover, the map $\sigma_n|H^n_a\colon H^n_a \to F^n_a$ is a resolution of the singularities of F^n_a, for all $a \in A$.*

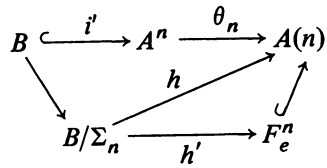
Proof. Since H^n is nonsingular, the same is true of H^n_1 by (18). But by Theorem 6, H^n_1 is isomorphic to the product variety $H^n_e \times A_1$; thus H^n_e is nonsingular by (17). By translation all of the fibers of φ_n are nonsingular, and the last statement follows from Theorem 4. □

Corollary 2. *Let A be an abelian variety of any dimension $q > 0$, and let $n > 0$ be such that $p \nmid n$. Then every fiber F^n_a of the map $f_n\colon A(n) \to A$ is a normal variety.*

Proof. Since A^n is normal (being nonsingular), the symmetric product is also normal [**12**, p. 58]. Arguing as in the proof of Corollary 1, we conclude that $A(n)_1$ and (hence) F^n_e are normal; thus all fibers are normal by translation.

Remark. The normality of the fibers of $f_n\colon A(n) \to A$ (assured when $p \nmid n$) has the consequence that these fibers are themselves quotient varieties for the action of a finite group.

Indeed, let B be the image of the embedding $i\colon A^{n-1} \to A^n$ used to define the surjective morphism $j\colon A^{n-1} \to F^n_e$ (15). Clearly B is (as set) the kernel of the homomorphism $A^n \to A$ given by addition, and is stable under the action of the symmetric group Σ_n on A^n. By restriction Σ_n acts on B, and we may form the quotient variety B/Σ_n. There is a commutative diagram



where h is the map resulting from the universal property of the quotient variety, since $\theta_n \circ i'$ is invariant under the action of Σ_n on B. The map h factors through the image F_e^n of $\theta_n \circ i'$ as shown; clearly h' is a bijection. We want to prove that h' is an isomorphism; this will be so if and only if h is an embedding (closed immersion).

As is pointed out in [**12**, p. 60], h need not be an embedding in general, unless the characteristic is zero or the group acts freely. However, since Σ_n does act freely on the open set of off-diagonal points of A^n (which B intersects nontrivially), it follows that h is an embedding on an open non-empty set. Consequently h' is a bijective birational morphism, and hence an isomorphism (by Zariski's Main Theorem) when F_e^n is normal. Thus if $p \nmid n$, the fibers of f_n: $A(n) \to A$ are quotient varieties, and the singularities being resolved in Corollary 1 are quotient singularities.

Example. Let A be an abelian surface in characteristic 0, and let $n = 2$. Then the map i: $A \to A^2$ sends $a \in A$ to the pair $(a, -a)$, and B is the set of all such pairs (notation as above). Thus $B/\Sigma_2 = F_e^2$ is the classical Kummer surface, created by identifying each point $a \in A$ with its inverse $-a$. The singular points of this surface correspond to the points of order 2 in A, and are resolved by being blown up into projective lines [**14**]. This is then a special case of Corollary 1, since the map $\sigma_2|H_e^2$: $H_e^2 \to F_e^2$ is an isomorphism except over the singular points of F_e^2, where the fiber is a projective line.

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